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Experimental determination of KPZ height-fluctuation distributions

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Abstract. Height-fluctuation distributions of nonequilibrium interfaces were analyzed using slow-combustion fronts propagating in sheets of paper. All distributions measured were definitely non-Gaussian. The experimental distributions for transient and stationary regimes were well fitted by the theoretical distributions proposed by Prähofer and Spohn in reference [9]. Consistent with the Galilean invariance of the system, the same distributions were found for horizontal fronts and, when determined along the normal to the slope, for fronts with a non-zero average slope.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 64.60.Ht Dynamic critical phenomena

1 Introduction

The dynamics of interfaces propagating in one space dimension, whose local velocity is in the direction of the local outward normal to the interface, provide typical examples of a class of nonequilibrium systems whose scaling properties are generally believed to be those of the Kardar-Parisi-Zhang (KPZ) [1] in 1+1 dimensions, $\partial_t h = \nu\partial_x^2 h + \frac{1}{2}\lambda(\partial_x h)^2 + \eta$. The temporal and spatial correlations in this class of nonequilibrium systems display power-law behaviors governed by scaling exponents $\beta = 1/3$ (or $z = \alpha/\beta$) and $\alpha = 1/2$, respectively [2–4]. Most of the experimental efforts on nonequilibrium interfaces have been devoted to finding one or more of these scaling exponents, see, e.g., references [3,16] and references therein. Other characteristic 'universal' features of nonequilibrium interfaces in the KPZ universality class, such as the universal amplitude ratios [5,3], the dimensionless 'coupling constant' at the KPZ fixed point [6,7], and their persistence properties [8], have received much less experimental interest (see, however, Refs. [17,18]), even though they are stringent tests of the scaling properties of the system.

In addition to the universal properties of nonequilibrium systems in the KPZ universality class, 'non-universal' properties of these systems have recently been discussed, especially those related to fluctuations of a propagating interface around its average position [9–12]. Distributions of these fluctuations appear to depend on the global geometry of the average interface, and they are also different in the transient and stationary regimes. The non-Gaussian nature of these distributions would provide yet another sensitive measure of the nonequilibrium nature of the dynamics of the interface. So far this possibility has not attracted experimental interest.

Exact results are now available for fluctuations of the KPZ type of interfaces [9], based on mapping of these fluctuations using the polynuclear growth (PNG) model as the starting point into a last-passage percolation problem. The statistics of the latter problem are, on the other hand, those of random permutations, and are thus related to distributions of the largest eigenvalues of certain ensembles of random matrices [13,14]. More specifically, if $h(x,t)$ is the position of the interface at point x at time t in a translationally invariant system, the probability distribution for local fluctuations of the position around its mean value is given by

$$P\Big(\frac{h(x,t_2) - h(x,t_1) - (t_2 - t_1)\langle\partial_t h\rangle}{A_q (t_2 - t_1)^{1/3}} \leq s\Big) = F_q(s), \quad (1)$$

where A_q is a constant depending on system-specific parameters, and $\langle\partial_t h\rangle$ is the average velocity of the interface in the time interval $t_1 < t < t_2$. The distribution function $F_q(s)$ depends on the global geometry of the average interface and also on its initial conditions: this distribution function is different for stationary and transient fluctuations meaning stationary and flat initial configurations of the interface, respectively.

For transient dynamics that evolve from initial conditions $h(x,t_1) \equiv 0$ for all x, realized for times $t_1 < t_2 < t_{\text{sat}}$ with t_{sat} the time at which the stationary sate is reached, the distribution function for fluctuation in the local position (height) of the interface is denoted by $F_1(s)$ in reference [9]. For a system in a stationary state, for which the

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times t_1 and t_2 defined above must satisfy $t_{\rm sat} < t_1 < t_2$, this distribution function is denoted by $F_0(s)$ [9]. Other geometries such as droplet, half droplet and half line were also considered, and the corresponding height-fluctuation statistics (for these geometries the statistics also depend on x) were found to be related to appropriate random-matrix ensembles [15]. Expressions for the distribution functions $F_q(s)$ and the respective probability densities $f_q(s) = dF_q(s)/ds$ in all these cases can be found in references. [9,13–15].

We have previously provided ample experimental evidence [16–20] that indicates the dynamics of slow-combustion fronts in paper asymptotically belong to the KPZ universality class. Based on this work, the nature of the non-asymptotic behavior of these fronts at short time and length scales is also rather well established. It would be quite natural now to find out, if in addition to the 'universal' KPZ behavior, also the 'non-universal' KPZ behavior described above is indeed realized in nonequilibrium interfaces. To this end we address here the question of height-fluctuation distributions in slow-combustion fronts, and determine in particular the distributions corresponding to the F_1 and F_0 distributions defined above.

Another question of some principal interest is the Galilean invariance of KPZ type of interfaces. This invariance means that interfaces with a non-zero average slope should behave the same way as the horizontal (on the average) interfaces. We therefore determine the height-fluctuation distributions also for slow-combustion fronts with a non-zero average slope. When the distributions are determined along the normal to the slope, they should coincide with the corresponding distributions for horizontal fronts. For completeness, we determine as well the λ parameter of the KPZ equation for these tilted fronts, the same way it was previously measured for horizontal fronts [17,19].

2 Experimental setup

The experimental setup used was the one reported earlier by Maunuksela et al. [16], and described in more detail in [17]. In brief, paper samples are made to smolder, with initially a linear combustion front, inside a chamber whose air flow can be controlled. The frame in which the samples are attached can be rotated freely so as to optimize the conditions for front propagation with respect to air flow. Propagating fronts are recorded with three parallel black and white CCD cameras each having 768 × 548 pixels. The images are joined together and saved on-line on a hard drive for further processing and data analysis. In the measurements reported here, the sample rate of the cameras was 2 images per second, i.e., $\Delta t = 0.5$ s, and the spatial resolution was 0.152 mm.

Copier paper of basis weight 80 g/m^2 and sample size 340 mm (width) by 600 mm was used in the experiments. To achieve flameless slow-combustion fronts, paper was impregnated with potassium nitrate that acted as an oxygen source. A dilute aqueous solution of KNO_3 was sprayed over the samples so as to prevent capillary flows that would induce systematic variations in the potassium-nitrate concentration within the sample [21]. Random small-scale variations in the concentration were obviously introduced by the spraying process. The average KNO_3 concentration of the samples varied between 1.3 and 1.9 g/m^2. After spraying the samples were dried in a press to maintain their planarity.

For a combustion-front experiment two sides of a sample were attached by pins to the frame in the combustion chamber. In these experiments the frame was placed vertically in the chamber in which the air flow was now directly upwards. Ignition of the front was made with a tungsten wire heated by an electric current. The wire was stretched over the sample with two metallic strings to keep its tension constant during heating, and to thereby get a linear front initially. The copier-paper samples were burned from top to bottom to minimize convective heat transfer during front propagation.

We used two initial conditions in the experiments: horizontal ignition and ignition with a slope of about ten degrees in the propagating fronts. For the latter ('tilted') ignition, the top end of the sample was cut parallel to the ignition wire to prevent position–dependent heat transfer from the induced combustion fronts. The tilt angle used was only selected because it was convenient for our three–camera system. The angle was also chosen such that the system was not expected to be near the known phase boundaries for related models in the same universality class [22]. For the horizontal and tilted ignition a total of 18 and 21 successful burns were recorded, respectively.

3 Simulations

We also studied the height-fluctuation distributions of simulated combustion fronts. The first simulation model used was the one presented in reference [20], i.e direct numerical solution of the discretized KPZ equation with realistic input noise. Input-noise matrices had previously been obtained by scanning β-radiographs of paper samples of an 80 g/m^2 copier paper [20]. There were only three 170 mm wide β-radiographs available, which made the simulation statistics quite limited. The noise-amplitude distributions were not symmetric as can be seen from Figure 1. In paper the inherent pore-size distribution produces a tail in the low-density limit of its density distribution, resulting in a high-amplitude tail in the structural noise as density and combustion speed are anticorrelated.

For horizontal ignition, simulations were carried out using both free and periodic boundary conditions. No noticeable difference was observed in the resulting height-fluctuation distributions. For tilted ignition we imposed free and fixed-average-slope boundary conditions. The former are close to the experimental conditions, and the corresponding simulations clearly show how boundary effects gradually straighten the front profile in a finite system. It is however evident from these simulations that there is a width of the sample around the middle line in which the

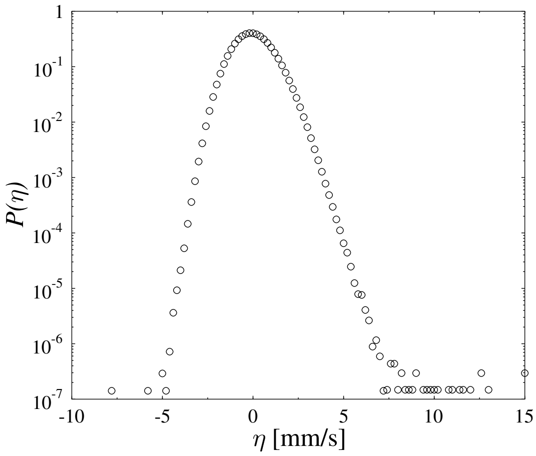

Fig. 1. Average noise-amplitude distribution of the scanned noise matrices that were first normalized to zero mean and unit variance. The skewness of the average distribution is 0.25.

local front-height fluctuations reach a saturated regime corresponding to a nonzero average slope.

The second model we used in the simulations was the polynuclear growth (PNG) model, which we used to study whether the finite system size has an effect on height-fluctuation distributions. The PNG model is a simple growth model in which 'nuclei' are deposited at random on an initially flat surface. The deposited nuclei then grow laterally with constant speed [4]. This model also belongs to the KPZ universality class. It does not require any input noise other than that related to random nucleation, so gathering enough statistics was not a problem. Once borders of two spreading nuclei touch each other they will coalesce. New nucleation events can take place on top of already existing layers of deposited matter.

We used a discrete version of the (1+1)-dimensional PNG model in our simulations. The deposited nuclei were one unit high and wide, and the lateral spreading rate was set to one lattice spacing per time step. Free and periodic boundary conditions were both used.

4 Results

It is not possible to maintain indefinitely a tilted slow-combustion front with free boundaries because the front will gradually straighten as a λ term responsible for KPZ-type behavior will drive small-tilt fluctuations inwards from the up-hill boundary. The leading edge of the front thus tends to get retarded from its constant-average-tilt position (see Fig. 2). These boundary effects will eventually penetrate the system and straighten out the whole front.

However, it is possible to study the behavior of the tilted part of a propagating front when an appropriate position window is applied in the analysis. There is obviously an upper limit to the time scale during which it is possible to examine in this way a tilted front. This time scale obviously depends on system size.

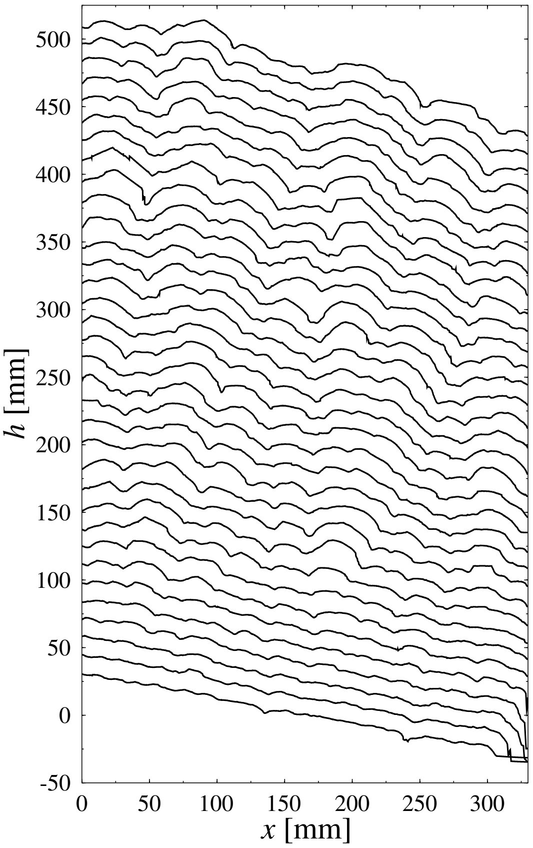

Fig. 2. A slow-combustion burn for a tilted ignition with an average slope of ten degrees. The time difference between the shown fronts is 20 s. Penetration inwards of boundary effects is clearly visible.

The (squared) front width is usually defined as $w^2(L,t) \equiv \langle\overline{(h-\bar{h})^2}\rangle$, where the overbar denotes spatial and the brackets noise averaging. For tilted fronts this definition is not quite consistent because the mean height $\bar{h}$ does not represent the 'average profile' of a front. For tilted fronts we therefore measured the front width with respect to linear least-squares fits of the fronts. The height-fluctuations of these fronts were measured in the direction normal to the initial front. For both tilted and horizontal fronts the height-fluctuation data from points close to the boundaries of the samples were excluded to avoid boundary effects [17].

In Figure 3 we show the measured height-fluctuation distribution for horizontal slow-combustion fronts in the transient ($w \sim t^{1/3}$) regime. Plotted also is a fit by the theoretical distribution f_1, where the original [9,13–15] horizontal scale was multiplied by a proper scaling factor, in this case by 0.67845. The scaling factor was selected such that the two distributions had equal variance after normalization. The mean values of the distributions were still different, and therefore the fitted distribution was shifted horizontally to fit the experimental one. We return to this point in the final section below.

A theoretical inversion of the measured distribution, shown in the inset of Figure 3, indicates that the

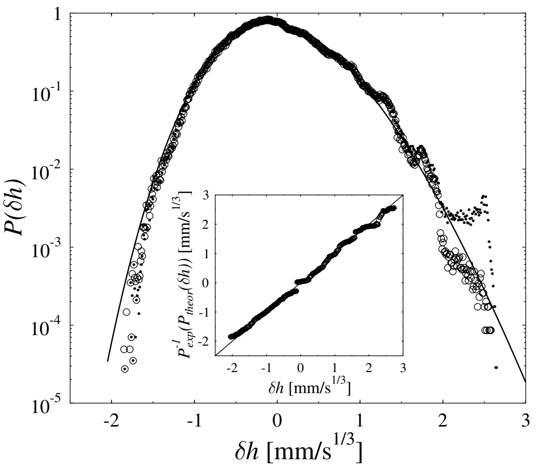

Fig. 3. Height-fluctuation distribution for horizontal fronts in the transient ($w \sim t^{1/3}$) regime, and a fit by a (scaled and shifted) theoretical distribution f_1. A theoretical inversion of the measured distribution is shown in the inset. The dots denote the measured data and the circles the data with an avalanche suppressed.

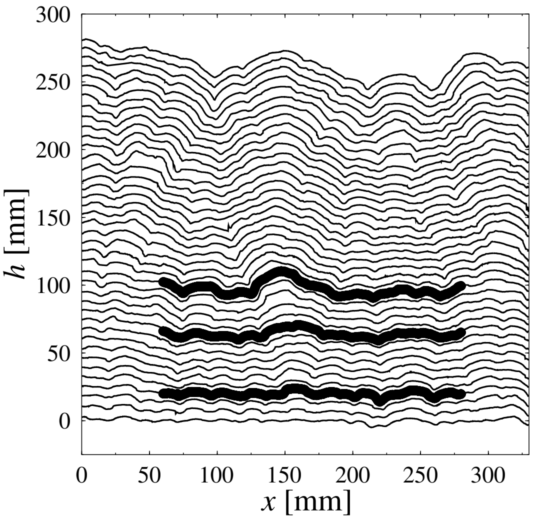

Fig. 4. The first half of a horizontal burn. The analysis window used in the transient regime is indicated by the lowest and highest of the thick lines. In this particular case a pronounced avalanche in the middle of the burn was removed by lowering the upper limit of the analysis window. The time difference between the fronts shown is 10 s.

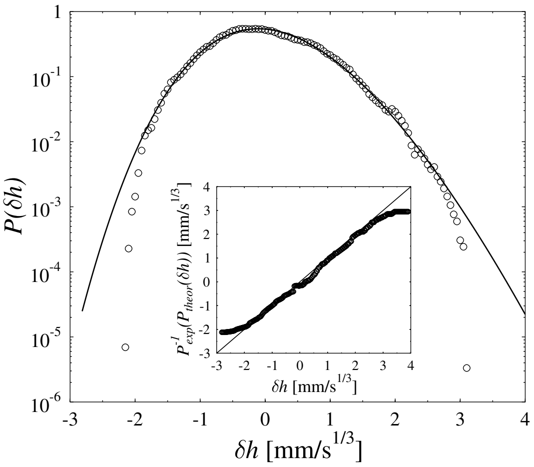

Fig. 5. Height-fluctuation distribution for tilted fronts in the transient regime, and a fit by a (scaled and shifted) f_1 distribution.

experimental result agrees quite well with the theoretical distribution except at large positive fluctuations where additional peaks appear. These peaks can be attributed to 'avalanches', i.e., to narrow intermittent parts of the front that rapidly advance ahead of the rest of it. This kind of behavior is triggered by quenched noise in the system (variations in the density of paper material and in the potassium-nitrate concentration), and would not be present in a pure KPZ system with uncorrelated white noise. In order to diminish the effect of such 'unwanted' avalanches on the height-fluctuation distribution, we suppressed one distinct avalanche from the measured data (see Fig. 4). The original raw data are shown by dots in Figure 3.

Figure 5 shows the height-fluctuation distribution measured for tilted fronts in the transient regime. These fluctuations are measured in the direction perpendicular to the tilted ignition front, i.e., the fronts of a tilted burn were rotated by 10 degrees before determining their height fluctuations. The skewness of the measured distribution is 0.33 while that of the f_1 distribution is 0.2935.

Height-fluctuation distributions were also determined for saturated fronts (saturated width w). This regime in front dynamics corresponds to stationary self-similar growth. Figure 6 shows the distribution measured for horizontal fronts together with a fit by the theoretical f_0 distribution. The skewness of the measured distribution is 0.32 while that of a f_0 distribution is 0.359.

It seems that, in the saturated regime, height fluctuations follow fairly closely the f_0 distribution. For large fluctuations, i.e. for large δh, our results fall below the theoretical distribution because of limited statistics and finite system size.

The results of PNG simulations proved to agree with theory already for quite small ($L = 2000$) system sizes. Figure 7 shows the height-fluctuation distribution for simulated fronts in the saturated regime together with a fit by a f_0 distribution. The simulation data consist of 100 individual runs for both boundary conditions. Only a small difference can be seen in the distributions for periodic and free boundary conditions. The excessive amount of fluctuations for $\delta h < 0$ in comparison with the theoretical result is a boundary effect, and is pronounced only for very small system sizes (as, e.g., for $L = 500$). For sample widths used in the experiments ($L \approx 2000$ pixels) this effect is detectable but not very pronounced when proper averaging is performed. We can thus expect that the sample width used in the experiments should not prevent us from seeing the theoretical distributions derived in the thermodynamic limit.

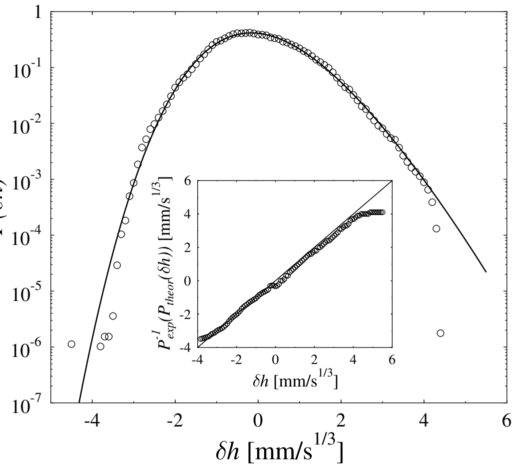

Fig. 6. Height-fluctuation distribution for horizontal fronts in the saturated regime, and a fit by a (scaled) f_0 distribution.

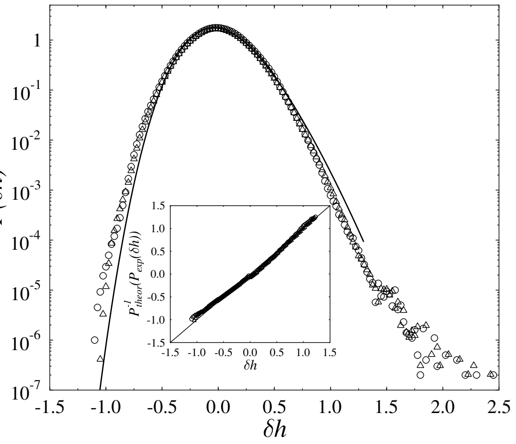

Fig. 7. Height-fluctuation distributions simulated for horizontal fronts in the PNG model in the saturated regime, and a fit by a (scaled) f_0 distribution. Circles and triangles denote the data for periodic and free boundary conditions, respectively.

Values for the λ parameter in the KPZ equation have previously been determined from fronts with horizontal ignition [19]. Because of Galilean invariance, the value of λ should be independent of the average slope of the fronts. It is thus interesting to directly check this invariance, as we now have data also for nonzero average slope. To this end we use the slope-dependent velocity of the fronts [19]: For a front that belongs to the KPZ universality class, the local velocity is given by $v(m) \approx c + \frac{\lambda}{2} m^2$, [2], where m is a nonzero average slope in an interval of length ℓ. From a parabolic fit to the present data for tilted fronts we find that λ=0.39(2) mm/s, in good agreement with the result 0.37(3) mm/s reported for horizontal fronts in [19].

5 Discussion

The theoretical results reported for height-fluctuation distributions [9] were derived for infinite systems for which front width can grow without saturation. In real life the finite size of the system leads to a transient regime of finite duration before saturation sets in. Thus there will always be a cutoff in the measured amplitude of fluctuations in the front position, and the data will fall below the theoretical distribution at large δh (for tilted fronts, Fig. 5). In addition, avalanches related to quenched noise in the system can produce large unwanted non-KPZ fluctuations in front heights (Fig. 3), and local pinning of the front can add to the large negative-fluctuation end of the distribution. In spite of these experimental artifacts, our results for front-height fluctuations support the theoretical predictions. The measured distributions are definitely not Gaussian, and their skewness values are not inconsistent, given the experimental uncertainty, with the theoretical predictions.

The average velocity $\langle \partial_t h \rangle$ in equation (1) was determined as a linear least-squares fit to the mean height as a function of time of the front. Within the indicated time interval, which was the interval over which the height-fluctuation averages were taken, these velocities varied in time. The front velocities averaged over the whole analysis intervals were also somewhat sample dependent. This caused the mean values of individual fluctuation distributions to differ from each other, and their direct summation was not meaningful. Because of this, the mean value was subtracted from every individual distribution before summation, and hence our experimental distributions have zero mean. This is why the theoretical f_1 distribution had to be transferred also to zero mean before fitting with it the transient height-fluctuation distributions. The numerical parameter left thereafter for comparing experiment with theory is the skewness of the distribution. The measured skewnesses were so large that there is no doubt the distributions are not Gaussian, but the skewnesses of the f_1 and f_0 distributions are so close to each others (0.2935 and 0.35941, respectively) that it is not easy to tell them apart within the present experimental accuracy. Fitting by f_1 the transient-regime and by f_0 the saturated-regime distributions gave, however, very consistent results.

As we only had three scanned images of large paper samples to be used as realistic noise in simulations of the KPZ equation, we did not have enough statistics to reliably simulate the height-fluctuation distributions in this way. These simulations could however be used to support the adopted way of analyzing the transient regime for fronts with nonzero average slope even when free boundary conditions were used. We used simulations of the PNG model to make sure that the experimental system size should not appreciably affect the theoretical height-fluctuation distribution (we used $L = 2000$ in these simulations while the measured sample width was composed of about 2000 pixels). It is evident that the finite width of the samples was not a problem when determining the experimental height-fluctuation distributions.

A result accomplished in this work worth noting here, related to the properties of KPZ fronts, was the Galilean invariance of these fronts. We were able to show that the height-fluctuation distributions are the same for horizontal fronts and for fronts with a non-zero average slope, provided that in the latter case the fluctuations are determined along the normal to the average front position. We also determined the λ parameter of the KPZ equation for these tilted fronts, and found it to be the same as the one determined previously [19] for horizontal fronts. This form of Galilean invariance of the system is yet another proof that slow-combustion fronts indeed belong to the universality class whose dynamics is governed by the KPZ equation.

In conclusion, we have demonstrated, by analyzing slow-combustion fronts propagating in sheets of paper, that the height-fluctuation distribution of stationary KPZ fronts is indeed given by the f_0 distribution of reference [9]. Likewise the height-fluctuation distribution of transient fronts is consistent with the f_1 distribution [9]. It is thus evident that there are differences between transient and stationary fluctuations in nonequilibrium systems. A similar difference has previously been suggested to exist in the persistence properties of these systems [8], but an attempt based on the same experimental system [18] was not successful in dealing then with the transient behavior. It would be instructive now to analyze also fronts with some other global geometry, e.g., circular fronts, for which yet another distribution should be found [9]. This problem is left as a challenge for future experimental efforts.

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